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A Study on Second-Order Linear Singular Systems using Leapfrog Method

S. Karunanithi, S. Chakravarthy and S. Sekar

Abstract— In this paper, we study the numerical method for solving second-order linear singular systems using Leapfrog method. And, we present two examples with initial condition [1] having two different solutions to illustrate the efficiency of the proposed method under Leapfrog method. Error graphs are presented to highlight the efficiency of the Leapfrog method.

Index Terms— Haar wavelet, Single term Haar wavelet series, Leapfrog method, singular systems, second order singular systems.

1 Introduction

One of the most important tasks in a study of dynamical systems is the numerical calculation of the trajectories. Thus far we have considered the integration method to be a black box into which we pop the system, initial conditions, method and time range and out pops a plot of the trajectories. Although this approach is common in courses on dynamical systems it obscures many of the pitfalls of numerical integration.

It is not possible at the present state of the art to choose a 'best' algorithm for the calculation of trajectories. There are several types of numerical algorithm, each with their own advantages and disadvantages. We shall consider a class of methods known as discrete variable methods. These methods approximate the continuous time dynamical system by a discrete time system. This means that we are not really simulating the continuous system but a discrete system which may have different dynamical properties. This is an extremely important point. [1-7, 16]

STHWS can have a significant impact on what is considered a practical approach and on the types of problems that can be solved. S. Sekar and team of his researchers [8 - 15] introduced the STHWS to study the time-varying nonlinear singular systems, analysis of the differential equations of the sphere, to study on CNN based hole-filter template design, analysis of the singular and stiff delay systems and nonlinear singular systems from fluid dynamics, numerical investigation of nonlinear volterra-hammerstein integral equations, to study on periodic and oscillatory problems, and numerical solution of nonlinear problems in the calculus of variations. In this paper, we consider the second order linear singular system to solve by using the Leapfrog method. The results are compared with STHWS method and with exact solution of the problem.

- *Department of Mathematics, Government Thirumagal Mills College, Gudiyattam – 632 602, Tamil Nadu, India E-mail:kap232008@gmail.com, s2005chak@yahoo.com*
- *Department of Mathematics, Government Arts College (Autonomous), Cherry road, Salem – 636 007, Tamil Nadu, India E-mail:sekar_nitt@rediffmail.com*

Hence, we use this Leapfrog method in the present paper to study second order linear singular systems with initial conditions. The organized paper is as follows: In Section 2, the Leapfrog method for solving second order linear singular systems is introduced. In Section 3 presents general form of second order linear singular systems. In Section 4, the Leapfrog and STHWS [1] method for solving second order linear singular systems of time-invariant and varying case is solved.

2 Leapfrog Method

In mathematics Leapfrog integration is a simple method for numerically integrating differential equations of the form $\ddot{x} = F(x)$, or equivalently of the form $\dot{v} = F(x), \dot{x} \equiv v$, particularly in the case of a dynamical system of classical mechanics. Such problems often take the form $\ddot{x} = -\nabla V(x)$, with energy function $E(x,v) = \frac{1}{2}|v|^2 + V(x)$, where V is the potential energy of the system. The method is known by different names in different disciplines. In particular, it is similar to the Velocity Verlet method, which is a variant of Verlet integration. Leapfrog integration is equivalent to updating positions $x(t)$ and velocities $v(t) = \dot{x}(t)$ at interleaved time points, staggered in such a way that they 'Leapfrog' over each other. For example, the position is updated at integer time steps and the velocity is updated at integer-plus-a-half time steps.

Leapfrog integration is a second order method, in contrast to Euler integration, which is only first order, yet requires the same number of function evaluations per step. Unlike Euler integration, it is stable for oscillatory motion, as long as the time-step Δt is constant, and $\Delta t \le 2/w$. In Leapfrog integration, the equations for updating position and velocity are

$$x_i = x_{i-1} + v_{i-1/2}\Delta t,$$
$$a_i = F(x_i)$$
$$v_{i+1/2} = v_{i-1/2} + a_i \Delta t,$$

where x_i is position at step $i, v_{i+1/2}$, is the velocity, or first

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derivative of *x,* at step $i+1/2, a_i = F(x_i)$ is the acceleration, or second derivative of *x,* at step *i* and Δt is the size of each time step. These equations can be expressed in a form which gives velocity at integer steps as well. However, even in this synchronized form, the time-step Δt must be constant to maintain stability.

$$x_{i+1} = x_i + v_i \Delta t + \frac{1}{2} a_i \Delta t^2,$$

$$v_{i+1} = v_i + \frac{1}{2}(a_i + a_{i+1})\Delta t.$$

One use of this equation is in gravity simulations, since in that case the acceleration depends only on the positions of the gravitating masses, although higher order integrators (such as Runge–Kutta methods) are more frequently used. There are two primary strengths to Leapfrog integration when applied to mechanics problems. The first is the time-reversibility of the Leapfrog method. One can integrate forward *n* steps, and then reverse the direction of integration and integrate backwards *n* steps to arrive at the same starting position. The second strength of Leapfrog integration is its symplectic nature, which implies that it conserves the (slightly modified) energy of dynamical systems. This is especially useful when computing orbital dynamics, as other integration schemes, such as the Runge-Kutta method, do not conserve energy and allow the system to drift substantially over time.

3 General Form of Second Order Linear Singular Systems

In this section, a time-invariant second order linear singular system of the form

$$K\ddot{x}(t) = A\dot{x}(t) + Bx(t) + Cu(t) \qquad (9)$$

with initial condition $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ is considered, where K is an $n \times n$ singular matrix, A and B are $n \times n$ and $n \times p$ constant matrices respectively. $x(t)$ is an n-state vector and $u(t)$ is the p-input control vector and C is an $n \times p$ matrix.

Also a time-varying second order singular system is represented by

$$K(t)\ddot{x}(t) = A(t)\dot{x}(t) + B(t)x(t) + C(t)u(t) \qquad (10)$$

with initial condition $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ where $K(t)$ is an $n \times n$ singular matrix, $A(t)$ and $B(t)$ are $n \times n$ and $n \times p$ matrices respectively. *C(t)* is an $n \times p$ matrix. The elements (not necessarily all the elements) of the matrices $K(t), A(t)$ and $B(t)$ are time dependent.

4 Example of Second Order Singular System of Time-Invariant Case

The second order linear time-invariant singular system of the form (9) and *B*=*C*=*0*, becomes $\ddot{x}(t) = A\dot{x}(t)$ with initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ [1]

$$\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \ddot{x} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \dot{x}$$

with initial conditions

$$x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \dot{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

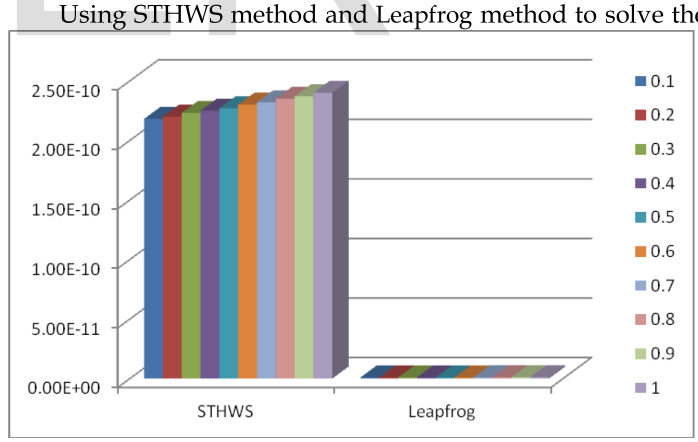
and the exact solution is

$$x_1(t) = 2.0$$
$$x_2(t) = -1 + 2\exp(t/2)$$

TABLE 1
Error Calculation of Time-invariant case for $x_2(t)$

t	$x_2(t)$	
	STHWS	Leapfrog
0.1	2.18E-10	1.01E-12
0.2	2.20E-10	1.02E-12
0.3	2.23E-10	1.03E-12
0.4	2.25E-10	1.04E-12
0.5	2.27E-10	1.05E-12
0.6	2.30E-10	1.06E-12
0.7	2.32E-10	1.07E-12
0.8	2.35E-10	1.08E-12
0.9	2.37E-10	1.09E-12
1	2.40E-10	1.10E-12

Fig. 1. Error graph for Time-invariant case for $x_2(t)$



Using STHWS method and Leapfrog method to solve the second order linear singular systems of time invariant case, the approximate and exact solutions for x_1 and x_2 are calculated for different values of time '*t*' and the error between them are shown in the Table 1 along with exact solutions. Error graphs are also presented in figure 1 shows the efficiency of Leapfrog.

5 Example of Second Order Singular System of Time-Varying Case

The following second order linear time-varying singular system of the form (10)[1]

$$\begin{bmatrix} 0 & 0 \\ 1 & t \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -t \\ 0 & -3 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ t^2+2t & 0 \end{bmatrix}\begin{bmatrix} e^t \\ 0 \end{bmatrix}$$

is considered with initial conditions $x(0)=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\dot{x}(0)=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

This problem can be rearranged as

$$\ddot{x}_1 = \dot{x}_1 - 3\dot{x}_2$$
$$\ddot{x}_2 = \dot{x}_2 + 2e^t(t+1)$$

The exact solution is

$$x_1(t) = 1 - t^3e^t$$
$$x_2(t) = t^2e^t$$
$$\dot{x}_1(t) = t^2e^t(-t-3)$$
$$\dot{x}_2(t) = te^t(t+2)$$

TABLE 2
ERROR CALCULATION OF TIME-VARYING CASE FOR $x_1(t)$

t	$x_1(t)$	
	STHWS	Leapfrog
0.1	2.64E-10	1.10E-12
0.2	2.61E-10	1.09E-12
0.3	2.59E-10	1.08E-12
0.4	2.57E-10	1.07E-12
0.5	2.54E-10	1.06E-12
0.6	2.52E-10	1.05E-12
0.7	2.49E-10	1.04E-12
0.8	2.47E-10	1.03E-12
0.9	2.44E-10	1.02E-12
1	2.42E-10	1.01E-12

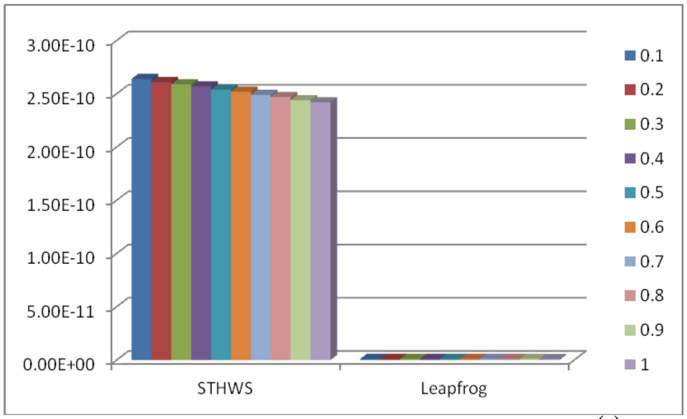


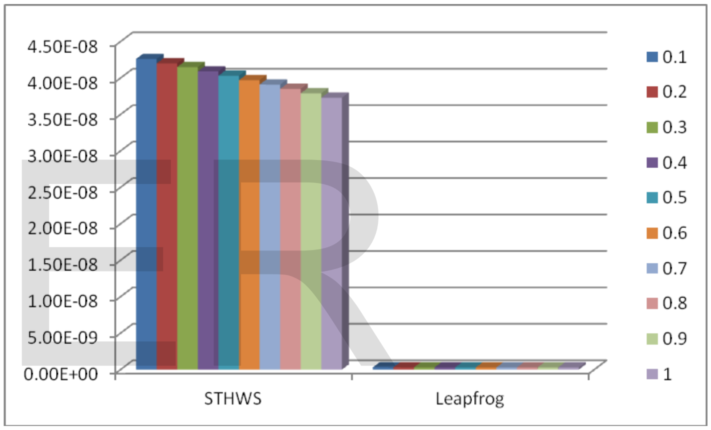
FIG. 2. ERROR GRAPH FOR TIME-VARYING CASE FOR $x_1(t)$

Using STHWS method and Leapfrog method to solve the second order linear singular systems of time varying case, the approximate and exact solutions for x_1 and x_2 are calculated for different values of time 't' and the error between them are shown in the Table 2 and 3 along with exact solutions. Also, error graphs are presented in figures 2 and 3.

TABLE 3
ERROR CALCULATION OF TIME-VARYING CASE FOR $x_2(t)$

t	$x_2(t)$	
	STHWS	Leapfrog
0.1	4.26E-8	3.05E-10
0.2	4.20E-8	3.04E-10
0.3	4.15E-8	3.03E-10
0.4	4.09E-8	3.02E-10
0.5	4.03E-8	3.01E-10
0.6	3.97E-8	2.99E-10
0.7	3.91E-8	2.98E-10
0.8	3.85E-8	2.97E-10
0.9	3.79E-8	2.96E-10
1	3.73E-8	2.95E-10

FIG. 3. ERROR GRAPH FOR TIME-VARYING CASE FOR $x_2(t)$



6 CONCLUSION

In this paper we introduce a new numerical method for solving second-order linear singular systems of time-invariant and time-varying cases. The efficiency and the accuracy of the Leapfrog method have been illustrated by suitable examples. The solutions obtained are compared well with the exact solutions of the second-order linear singular systems of time-invariant and time-varying cases and STHWS method [1]. It has been observed that the solutions by our method show good agreement with the exact solutions. From the numerical examples, we could conclude that the proposed Compare to STHWS method, Leapfrog method almost coincides with the exact solution of the second-order linear singular systems of time-invariant and time-varying cases [1] (refer Table 1 - 3 and Figure 1 - 3).

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